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Higher-Order Susceptibilities of the Regular and the Random Ising Model on the Cayley Tree. I

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Explicit expressions for the fourth-order susceptibility $\chi^{(4)}$, the fourth derivative of the *bulk* free energy with respect to the external field, are given for the regular and the random-bond Ising model on the Cayley tree in the thermodynamic limit, at zero external field. The fourth-order susceptibility for the regular system diverges at temperature $T_c^{(4)} = 2k_B^{-1}J/\ln\{1 + 2/[(z-1)^{3/4} - 1]\}$, confirming a result obtained by Müller-Hartmann and Zittartz [*Phys. Rev. Lett.* **33**:893 (1974)]; Here z is the coordination number of the lattice, J is the exchange integral, and k_B is the Boltzmann constant. The temperatures at which $\chi^{(4)}$ and the ordinary susceptibility $\chi^{(2)}$ diverge are given also for the random-bond and the random-site Ising model and for diluted Ising models.

KEY WORDS: Cayley tree; Ising model; higher-order susceptibilities; critical temperature; phase transition; random Ising model; diluted Ising model; critical concentration.

1. INTRODUCTION

Matsuda[1] and von Heimburg and Thomas[2] showed that the susceptibility at zero external field of the regular Ising model on the Cayley tree diverges at $T_c^{(2)} \equiv 2k_B^{-1}J/\ln\{1 + 2/[(z-1)^{1/2} - 1]\}$, a temperature lower than the critical temperature of the Bethe lattice $T_B \equiv k_B^{-1}J/\ln[z/(z-2)]$, where the *local* susceptibility at a site near the center of the Cayley tree diverges. Here J is the exchange integral, z is the coordination number of the lattice, and k_B is the Boltzmann constant. This fact was confirmed by the explicit expressions for the susceptibility.[2,3]

Müller-Hartmann and Zittartz[4] investigated the *bulk* free energy of the regular system as a function of the external field and showed that its

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$(2n)$th derivative with respect to the external field, $\chi^{(2n)}$ diverges at $T_c^{(2n)} = 2k_B^{-1}J/\ln\{1+2/[(z-1)^{1-1/2n}-1]\}$ at zero external field, for $n=1, 2, \ldots$; we shall call $\chi^{(2n)}$ the $(2n)$th susceptibility. Falk[5] obtained an upper bound to the nonpositive quantity $\chi^{(4)}$ and showed that the upper bound diverges to *minus* infinity for all $0 < T < T_c^{(4)}$;[2] notation $u^{(4)}$ is often used in place of $-\chi^{(4)}$. For the diluted Ising model, Heinrichs[6] studied the higher-order susceptibilities at low temperatures and showed that the critical concentration $p_c^{(2n)}$ at which $T_c^{(2n)} = 0$ is equal to $p_c^{(2n)} = (z-1)^{-(2n-1)/2n}$.[3] Gonçalves da Silva[7] studied the divergence of $\chi^{(2n)}$ for the random-bond Ising model with equal probabilities of exchange integrals J and $-J$. Horiguchi and Morita[8] discussed the divergence of $\chi^{(2n)}$ for the general random-bond Ising model. The purpose of the present series of papers is to give an explicit expression of the zero-field value of the $(2n)$th susceptibility for the regular as well as for the random Ising model on the Cayley tree, to confirm Müller-Hartmann and Zittartz's result for the regular system, to confirm Heinrichs' result for the diluted system, and to give the temperatures when $\chi^{(2n)}$ diverges also for the random systems.

In the present series of papers, we study the properties of the whole system on the Cayley tree at the thermodynamic limit, and do not discuss the properties at the central part of the Cayley tree. We consider Cayley trees of coordination number z with $(N+1)$ shells (generations) and take the limit $N \to \infty$. We have only one site on the zeroth shell. We call the site the zeroth site; see Fig. 1. That site is assumed here to have only $B = z-1$ nearest neighbors, and then the total number of the sites in the system N_s is given by

$$N_s = \frac{B^{N+1}-1}{B-1} \tag{1.1}$$

For the Ising model on a Cayley tree of $(N+1)$ shells, we denote the free energy of the system under an external field h by $F(N,h)$: For the regular system it is given by

$$-\beta F(N,h) = \ln \mathrm{Tr}\left[\exp(-\beta H)\right] \tag{1.2}$$

where H is the Hamiltonian of the system given by

$$H = -h\sum_i \sigma_i - \sum_{\substack{i>j \\ (i,\,j:\ \text{n.n.})}} \sum J_{ij}\sigma_i\sigma_j, \tag{1.3}$$

[2] In (28) of his paper [5] the symbol $\leqslant$ was omitted by the printer and should appear just after $\beta^{-3}D_4$.

[3] According to reference 6, the odd-order susceptibilities $\chi^{(2n+1)}$ also diverges at the concentration $p_c^{(2n+1)} = (z-1)^{-2n/(2n+1)}$ for $n = 1, 2, \ldots$, at zero temperature. But the odd-order susceptibilities $\chi^{(2n+1)}$ are quantities which involve an average of a product of an odd number of spins, and are zero for the present system in zero external field.

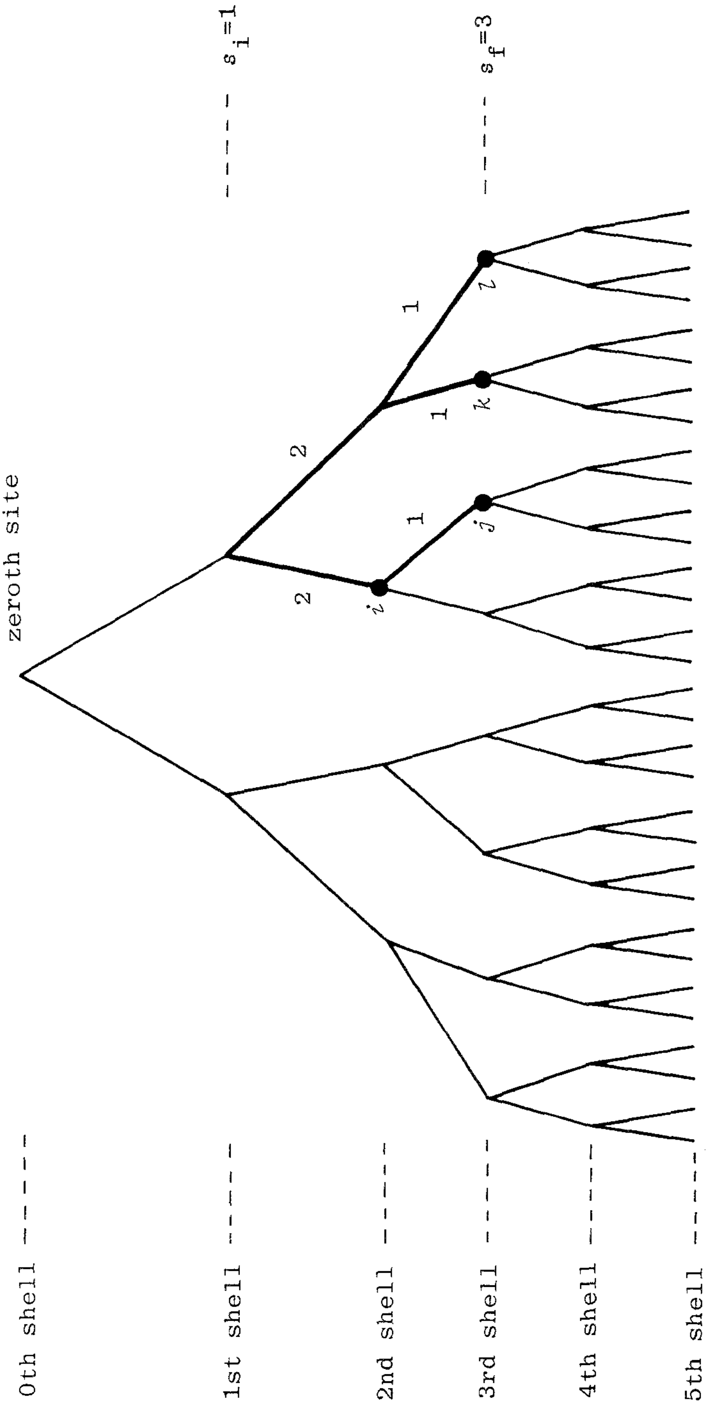

Fig. 1. A finite Cayley tree of $B = z - 1 = 2$, six shells $(N + 1 = 6)$, and a diagram for a contribution to $\chi^{(4)}(6, 0)$.

and $J_{ij}=J$. Here σ_i is the spin variable, taking on ± 1, for the site i, and $(i, j$: n.n.) denotes that the sites i and j are nearest neighbors of each other. $\beta = 1/k_B T$ and T is the absolute temperature. We define the nth susceptibility per site $\chi^{(n)}(N,h)$ of this system by

$$\chi^{(n)}(N,h) = \frac{\beta^{n-1}}{N_s} \frac{\partial^n[-\beta F(N,h)]}{\partial(\beta h)^n} \tag{1.4}$$

We then define the thermodynamic limits $\chi^{(n)}$ and $\chi_T^{(n)}$ by

$$\chi^{(n)} = \lim_{N\to\infty} \chi^{(n)}(N, h=0) \tag{1.5}$$

$$\chi_T^{(n)} = \lim_{h\to 0} \lim_{N\to\infty} \chi^{(n)}(N,h) \tag{1.6}$$

For random systems, J_{ij} in (1.3) are random numbers governed by a distribution function, and the quantity on the right-hand side of (1.2) must be averaged with respect to the distribution functions of $\{J_{ij}\}$.

In Section 2 of the present paper, we calculate the fourth-order susceptibility $\chi^{(4)}$ for the regular system. We consider the second or ordinary susceptibility $\chi^{(2)}$ in Section 3. We discuss $\chi^{(4)}$ and $\chi^{(2)}$ for the random-bond and the random-site Ising model and for the diluted Ising model in Sections 4 and 5. Section 6 is a summary. A proof of the equality $\chi^{(n)} = \chi_T^{(n)}$ for general n and the calculation of the higher-order susceptibilities $\chi^{(2n)}$ are left for separate papers of this series.

2. FOURTH-ORDER SUSCEPTIBILITY

In the present section, we focus on the fourth-order susceptibility, the fourth derivative of the free energy with respect to the external field, for a regular Ising model. For a finite system, it is expressed as follows:

$$\chi^{(4)}(N,0) = \frac{\beta^3}{N_s} \sum_i \sum_j \sum_k \sum_l [\langle\sigma_i\sigma_j\sigma_k\sigma_l\rangle - \langle\sigma_i\sigma_j\rangle\langle\sigma_k\sigma_l\rangle - \langle\sigma_i\sigma_k\rangle\langle\sigma_j\sigma_l\rangle - \langle\sigma_i\sigma_l\rangle\langle\sigma_j\sigma_k\rangle] \tag{2.1}$$

at zero external field. Here $\langle A\rangle$ denotes

$$\langle A\rangle = \mathrm{Tr}\, e^{-\beta H_0} A / \mathrm{Tr}\, e^{-\beta H_0}$$

$$H_0 = -\sum_{\substack{i>j \\ (i,\, j:\ \text{n.n.})}} \sum J_{ij}\sigma_i\sigma_j,$$

where $J_{ij}=J$ for a regular system.

The average $\langle\sigma_i\sigma_j\rangle$ is expressed by the diagram composed of two vertices at the sites i and j and the bonds on the route from one of the sites i to the other j on the lattice, and the value of $\langle\sigma_i\sigma_j\rangle$ is equal to the product of factors $w \equiv \tanh(\beta J_{kl})$ for the bonds (kl) on the diagram. The product $\langle\sigma_i\sigma_j\rangle\langle\sigma_k\sigma_l\rangle$ is represented by the diagram obtained by superposing the diagram for $\langle\sigma_i\sigma_j\rangle$ on the diagram for $\langle\sigma_k\sigma_l\rangle$. In this diagram, the superposed two parts may be either disconnected, or have one point in common, or have in common a number of subsequent bonds, which we call double bonds. For the three terms $\langle\sigma_i\sigma_j\rangle\langle\sigma_k\sigma_l\rangle$, $\langle\sigma_i\sigma_k\rangle\langle\sigma_j\sigma_l\rangle$, and $\langle\sigma_i\sigma_l\rangle\langle\sigma_j\sigma_k\rangle$, there occur the following two alternatives: (i) The diagrams for the three terms are all connected and equal to each other, having no double bonds. Each of the three have the same contribution as $\langle\sigma_i\sigma_j\sigma_k\sigma_l\rangle$, and is called the diagram for the vertices at i, j, k, l. (ii) One of them is disconnected and cancels the contribution of $\langle\sigma_i\sigma_j\sigma_k\sigma_l\rangle$ in (2.1) exactly. The other two are connected, with double bonds, and each has the same contribution. We call this diagram with double bonds the diagram for the vertices at i, j, k, l in this case; see Fig. 1 for an example. By using these observations and the definition of the diagrams for the vertices at i, j, k, l, (2.1) is reduced to

$$\chi^{(4)}(N,0) = -2\frac{\beta^3}{N_s}\sum_i\sum_j\sum_k\sum_l[\text{the diagram for the vertices at } i,j,k,l] \tag{2.2}$$

In calculating the sum in (2.2), we first note that (i) the same diagram with four fixed sites for the vertices appears repeatedly for all the different ways in which labels i, j, k, l are associated to the four sites for the vertices.

Let us consider a diagram in the summand. Let us assume that the vertices and the edges of the bonds constituting the diagram are on the shells from the s_ith to the s_fth. Only one point on the s_ith shell is occupied by a vertex or an edge of a bond of the diagram; see Fig. 1 for an example. We call that site the top of the diagram. We have topologically equivalent diagrams, having the top at all the different sites within $[N-(s_f-s_i)]$ shells. Hence we have $[B^{N-(s_f-s_i)+1}-1]/[B-1]$ equal contributions, so that we have the contribution

$$\frac{B^{N-(s_f-s_i)+1}-1}{B-1}[\text{the diagram with a fixed top position}]$$

to the sum in (2.2). Taking account of the factor $1/N_s$, the contribution to $-\chi^{(4)}/2\beta^3$ in the limit of $N\to\infty$ is

$$B^{-(s_f-s_i)}[\text{the diagram with a fixed top position}] \tag{2.3}$$

In order to account for the factor $B^{-(s_f-s_i)}$, (ii) factor B^{-1} is associated to each shell to which a bond of the diagram belongs.

We classify the diagrams with a fixed top position by the structure between the s_ith shell involving the top position and the $(s_i + 1)$th shell. Figure 2 shows all the possible structures. On each structure, the numeral above the line for the s_ith shell is the number of vertices which coincide with the top position. The numeral 1 or 2 associated to each line between the two shells denotes the number of vertices connected to the line on the $(s_i + 1)$th shell or below it, and also it shows that the line is a simple or double bond, accordingly. The set of numerals, 3, 1, denotes that the number of vertices connected to the line on the $(s_i + 1)$th shell or below it is three and the line is a simple bond. The number of ways of drawing one, two, three, and four bonds from the top site, are B, $B(B-1)$, $B(B-1)(B-2)$, and $B(B-1)(B-2)(B-3)$, respectively. We have a factor B^{-1} stated above in addition. When we have 1, 2, 3, and 4 vertices on the s_ith shell, we associate i, j, k, l to it in $\binom{4}{1}$, $\binom{4}{2}$, $\binom{4}{3}$, and $\binom{4}{4}$ ways, respectively. Then we write $\chi^{(4)}$ as follows:

$$\chi^{(4)} = -\tfrac{1}{2}\beta^3 g(4) \tag{2.4}$$

$$\begin{aligned} g(4) = 4\Bigg[&\binom{4}{4} + \binom{4}{3} w f(1) + \binom{4}{2} w_2 f(2) + \binom{4}{1} w_1 f(3) \\ &+ \binom{4}{2}(B-1)w^2 \frac{1}{2!} f(1,1) + \binom{4}{1}(B-1) w w_2 f(2,1) \\ &+ (B-1) w w_1 f(3,1) + (B-1) w_2^2 \frac{1}{2!} f(2,2) \\ &+ \binom{4}{1}(B-1)(B-2) w^3 \frac{1}{3!} f(1,1,1) \\ &+ (B-1)(B-2) w^2 w_2 \frac{1}{2!} f(2,1,1) \\ &+ (B-1)(B-2)(B-3) w^4 \frac{1}{4!} f(1,1,1,1)\Bigg] \end{aligned} \tag{2.5}$$

Here w_2 and w_1 are the factors for the lines labeled by 2 and (3, 1), respectively, in Figs. 2 and 3. Within the present section, w_2 and w_1 are put equal to w^2 and w, respectively:

$$w_2 = w^2, \qquad w_1 = w \tag{2.6}$$

The factors $1/2!$, $1/2!$, $1/2!$, $1/3!$, and $1/4!$ are placed in front of $f(1,1)$, $f(2,2)$, $f(2,1,1)$, $f(1,1,1)$, and $f(1,1,1,1)$ respectively, so that we can calculate their contributions regarding the lines with the same number as distinctive. The diagrams for $f(1)$, $f(2)$, $f(1,1)$, ... on the sth shell are classified by the structures between the sth and $(s+1)$th shells as in Fig. 3.

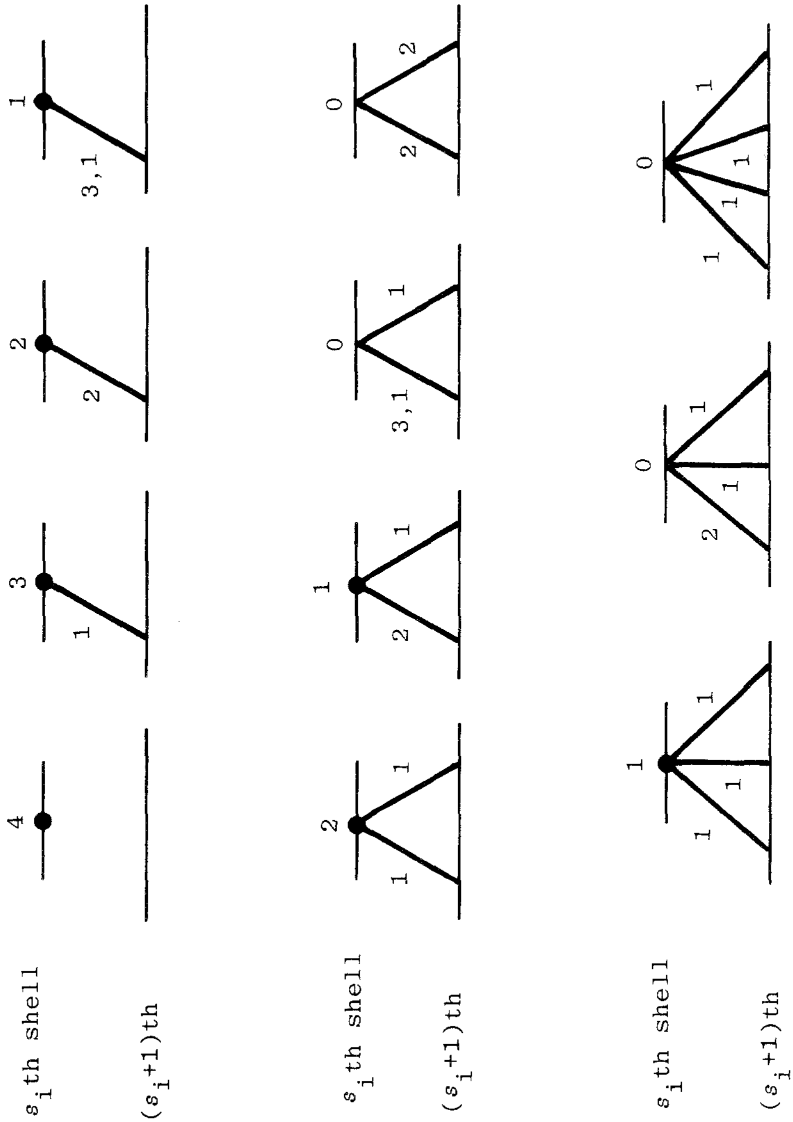

Fig. 2. The classification of the contributions to $g(4)$.

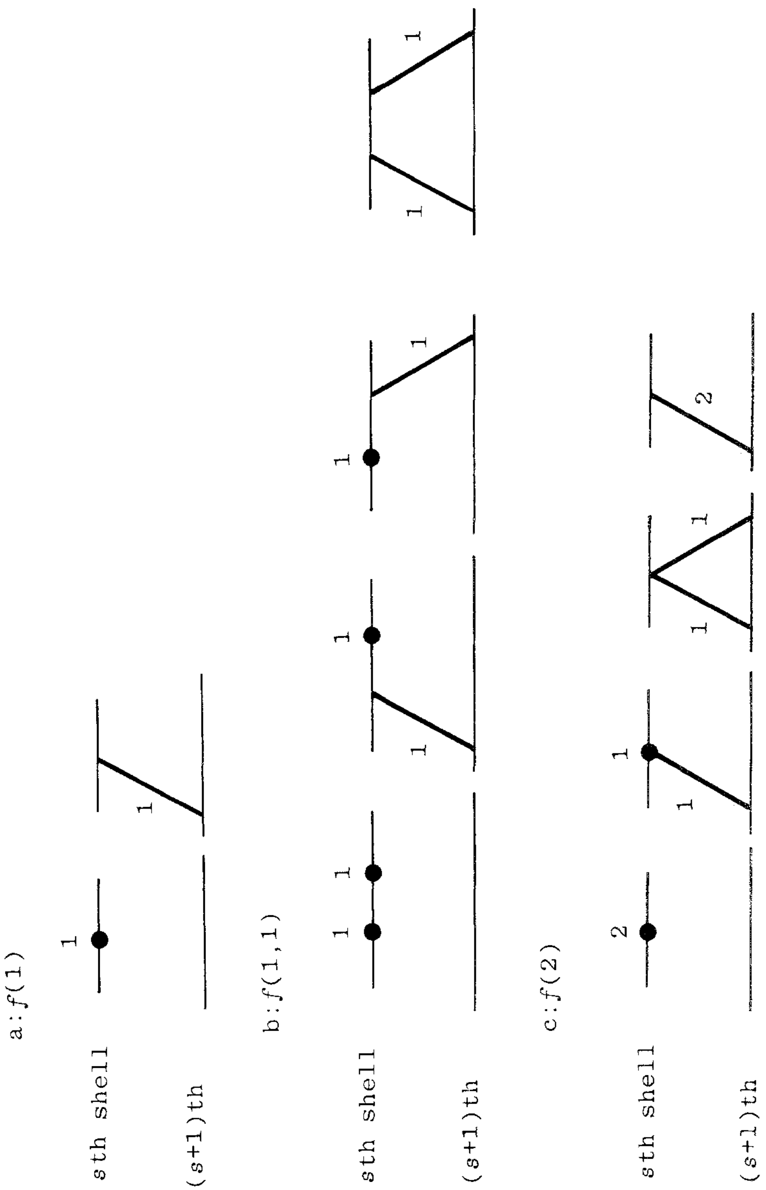

Fig. 3. The classification of the contributions to (a) $f(1)$, (b) $f(1, 1)$, (c) $f(2)$.

They give

$$f(1) = 1 + wf(1) \tag{2.7a}$$

$$f(1,1) = 2 + 2\times 2wf(1) + Bw^2 f(1,1) \tag{2.7b}$$

$$f(2) = 1 + \binom{2}{1} wf(1) + (B-1)w^2 \frac{1}{2!} f(1,1) + w_2 f(2) \tag{2.7c}$$

$$\cdots$$

Solving these for $f(1), f(1,1), f(2), \ldots,$ we have

$$f(1) = \frac{1}{1-w} \tag{2.8a}$$

$$f(1,1) = \frac{1}{1-Bw^2}\left[2 + 2\times 2wf(1)\right] \tag{2.8b}$$

$$f(2) = \frac{1}{1-w_2}\left[1 + \binom{2}{1} wf(1) + (B-1)\times \frac{w^2}{2!} f(1,1)\right] \tag{2.8c}$$

$$f(1,1,1) = \frac{1}{1-B^2w^3}\left[3! + 3\times 2\times 3wf(1) + 3\times 3Bw^2 f(1,1)\right] \tag{2.8d}$$

$$f(2,1) = \frac{1}{1-Bww_2}\left\{3 + (3+3\times 2)wf(1) + 3w_2 f(2) + \left[3B + 3(B-1)\frac{1}{2!}\right]\right.$$
$$\left.\times w^2 f(1,1) + B(B-1)w^3 \frac{1}{2!} f(1,1,1)\right\} \tag{2.8e}$$

$$f(3) = \frac{1}{1-w_1}\left[1 + 3wf(1) + 3w_2 f(2) + 3(B-1)w^2\frac{1}{2!} f(1,1)\right.$$
$$\left. + (B-1)ww_2 f(2,1) + (B-1)(B-2)w^3\frac{1}{3!} f(1,1,1)\right] \tag{2.8f}$$

$$f(1,1,1,1) = \frac{1}{1-B^3w^4}\left[4! + 4!\times 4wf(1) + 4\times 3\binom{4}{2}Bw^2 f(1,1)\right.$$
$$\left. + 4\times 4B^2w^3 f(1,1,1)\right] \tag{2.8g}$$

$$f(2,1,1) = \frac{1}{1-B^2w^2w_2}\left\{4\times 3 + (2\times 4\times 3 + 4!)wf(1) + 4\times 3w_2 f(2)\right.$$
$$+ \left[4\times 3(B-1)\frac{1}{2!} + 4!B + \binom{4}{2}B\right]$$
$$\times w^2 f(1,1) + 4\times 2Bww_2 f(2,1)$$
$$+ 4B\left[2(B-1)\frac{1}{2!} + B\right]w^3$$
$$\left.\times f(1,1,1) + \frac{1}{2!}B^2(B-1)w^4 f(1,1,1,1)\right\} \tag{2.8h}$$

$$f(3,1) = \frac{1}{1 - Bww_1} \left\{ 4 + (4 + 4 \times 3)wf(1) + 4 \times 3w_2 f(2) + 4w_1 f(3) \right.$$

$$+ \left[4 \times 3(B-1)\frac{1}{2!} + \binom{4}{2} B \right] w^2 f(1,1)$$

$$+ \left[4B + 4(B-1) \right] ww_2 f(2,1)$$

$$+ \left[4(B-1)(B-2)\frac{1}{3!} + 4 \times (B-1)\frac{1}{2!} \right] w^3 f(1,1,1)$$

$$+ B(B-1)w^2 w_2 f(2,1,1)$$

$$\left. + B(B-1)(B-2)\frac{1}{3!} w^4 f(1,1,1,1) \right\} \tag{2.8i}$$

$$f(2,2) = \frac{1}{1 - Bw_2^2} \left\{ \binom{4}{2} + 4 \times 3 \times 2wf(1) + \binom{4}{2} \times 2w_2 f(2) \right.$$

$$+ \left[4 \times 3B + \binom{4}{2} \times 2(B-1)\frac{1}{2!} \right] w^2 f(1,1)$$

$$+ 4 \times 2Bww_2 f(2,1) + 4 \times 2B(B-1)\frac{1}{2!} w^3 f(1,1,1)$$

$$+ B(B-1)w^2 w_2 f(2,1,1)$$

$$\left. + \left(\frac{1}{2!}\right)^2 B(B-1)^2 w^4 f(1,1,1,1) \right\} \tag{2.8j}$$

These equations are solved from the top equation successively. The functions $f(1, 1)$, $f(1, 1, 1)$, and $f(1, 1, 1, 1)$ are obtained from $f(1)$ by iterations between themselves:

$$f(1,1) = \frac{2(1+w)}{(1-w)(1-Bw^2)} \tag{2.9a}$$

$$f(1,1,1) = \frac{3!(1 + 2w + 2Bw^2 + Bw^3)}{(1-w)(1-Bw^2)(1-B^2w^3)} \tag{2.9b}$$

$$f(1,1,1,1) = \frac{4!(1 + 3w + 5Bw^2 + 3Bw^3 + 3B^2w^3 + 5B^2w^4 + 3B^3w^5 + B^3w^6)}{(1-w)(1-Bw^2)(1-B^2w^3)(1-B^3w^4)} \tag{2.9c}$$

If we use the relations (2.6), the other quantities $f(2), f(2,1), \ldots$ are expressed in terms of these by simple relations, which are given in the Appendix. In particular, $g(4)$ is given by

$$-\frac{1}{2}\beta^{-3}\chi^{(4)} = \frac{1}{4}g(4) = \left[\frac{1}{4!}(1-w^4) + \frac{(B-1)w^2(1-w^2)}{6(1-Bw^2)}\right]f(1,1,1,1) + \frac{2w(1+w)(1-w^2)}{3(1-Bw^2)}f(1,1,1) \tag{2.10}$$

Now we see that $g(4)$ is expressed as a sum of products of $1/(1-B^n w^m)$ with $n < m \leqslant 4$. When w is increased from zero, the factor $1/(1-B^3w^4)$ first diverges. The coefficients are all positive and hence there is no possibility that the coefficients of $1/(1-B^3w^4)$ cancel. Thus we see that $\chi^{(4)} = -\frac{1}{2}\beta^3 g^{(4)}$ really diverges when $B^3w^4 = 1$.

3. SUSCEPTIBILITY

Before going to random systems in next sections, we consider the ordinary susceptibility per site:

$$\chi^{(2)}(N,0) = \frac{\beta}{N_s}\sum_i \sum_j \langle \sigma_i \sigma_j \rangle \tag{3.1}$$

at zero external field. We now have diagrams with two vertices. We calculate (3.1) by classifying the diagrams by a fixed top position. The diagrams for $\beta^{-1}\chi^{(2)} = g(2)$ are classified into three diagrams, which are the first three given in Fig. 3c. Thus we get

$$\beta^{-1}\chi^{(2)} = g(2) = 1 + 2wf(1) + (B-1)w^2\frac{1}{2!}f(1,1) \tag{3.2}$$

By substituting from (2.8a) and (2.8b), we obtain

$$\chi^{(2)} = \frac{\beta(1+w)^2}{1-Bw^2} \tag{3.3}$$

a result given previously.[2,3]

4. RANDOM-BOND ISING MODEL AND DILUTED ISING MODELS

We now consider the random system for which the exchange integrals J_{ij} for bond ij are random numbers governed by a distribution function, independently of the values of J_{kl} on other bonds. The calculation for that case proceeds in a similar way to the preceding sections. The only difference is that we have an average with respect to the distribution of $\{J_{ij}\}$ on the right-hand side of (2.1) and, as a consequence, w and w_v must be replaced by their averages in (2.5), (2.7), and (3.2) and hence in (2.8), (2.9), and (3.3); so that we put

$$w = \langle \tanh(\beta J) \rangle_c \tag{4.1}$$

$$w_\nu = \langle \tanh^\nu(\beta J) \rangle_c \tag{4.2}$$

where $\langle A(J) \rangle_c$ for an arbitrary function $A(J)$ of J denotes

$$\langle A(J) \rangle_c = \int A(J) P(J)\, dJ \tag{4.3}$$

where $P(J)$ is the distribution function of J for each bond.

Now $\chi^{(4)}$ is given by (2.4), (2.5), and (2.8). Some of (2.8) may be replaced by (2.9). The factor which diverges at the highest temperature is either $1/(1 - B^3 w^4)$ or $1/(1 - B w_2^2)$. In a preceding note,[8] we define functions $\beta_2(x)$ and $\beta_1(x)$ by denoting the positive solution β of $\langle \tanh^2(\beta J) \rangle_c = x$ by $\beta_2(x)$ and the smallest positive solution β of $|\langle \tanh(\beta J) \rangle_c| = x$ by $\beta_1(x)$. If we denote

$$\beta_1^{(4)} = \beta_1(B^{-3/4}), \qquad \beta_2^{(4)} = \beta_2(B^{-1/2}) \tag{4.4}$$

and then

$$\beta_c^{(4)} = \min(\beta_1^{(4)}, \beta_2^{(4)}) \tag{4.5}$$

the highest temperature $T_c^{(4)}$ at which a term in the $\chi^{(4)}$ diverges is given by

$$T_c^{(4)} = 1/k_B \beta_c^{(4)} \tag{4.6}$$

The ordinary susceptibility $\chi^{(2)}$ is given by (3.3) with (4.1). It diverges at $T_c^{(2)} = 1/k_B \beta_c^{(2)}$, where $\beta_c^{(2)}$ is given by

$$\beta_c^{(2)} = \beta_1^{(2)} = \beta_1(B^{-1/2}) \tag{4.7}$$

In the special case of diluted-bond Ising model, where the probabilities of the exchange integral to be J and 0 are p and $1 - p$, respectively, w and w_ν are

$$w = p \tanh(\beta J), \qquad w_\nu = p \tanh^\nu(\beta J) \tag{4.8}$$

In this case, $\chi^{(4)}$ diverges at $T = T_c^{(4)} = 1/k_B \beta_c^{(4)}$ when $B^3 w^4 = 1$; it is given by

$$B^3 p^4 \tanh^4(\beta_c^{(4)} J) = 1 \tag{4.9}$$

The critical concentration $p_c^{(4)}$ at which $T_c^{(4)} = 0$, is given by

$$p_c^{(4)} = B^{-3/4} \tag{4.10}$$

In the diluted-site Ising model where the probability of the magnetic site is p, and the exchange integral between a nearest-neighbor pair of magnetic sites is J, the fourth-order susceptibility $\chi_{r\text{-}s}^{(4)}$ is obtained from the corresponding one $\chi^{(4)}_{r\text{-}b}$ for the diluted-bond Ising model considered in the preceding paragraph, by multiplying p as a factor for the top site on the s_ith shell:

$$\chi^{(4)}_{r\text{-}s} = p\chi^{(4)}_{r\text{-}b} \tag{4.11}$$

In a succeeding paper, we show that $\chi^{(2n)}_{r\text{-}s} = p\chi^{(2n)}_{r\text{-}b}$ in general. The critical temperature $T_c^{(4)}$ and the critical concentration $p_c^{(4)}$ are given by (4.9) and (4.10), respectively. Equation (4.10) for this case was obtained by Heinrichs.[6]

The ordinary susceptibility $\chi^{(2)}_{r\text{-}b}$ for the diluted-bond system is given by

$$\chi^{(2)}_{r\text{-}b} = \frac{\beta[1 + p\tanh(\beta J)]^2}{1 - Bp^2\tanh^2(\beta J)} \tag{4.12}$$

and then $\chi^{(2)}_{r\text{-}s}$ is equal to $p\chi^{(2)}_{r\text{-}b}$. This expression for $\chi^{(2)}_{r\text{-}s}$ is consistent with Heinrichs' equation (36a)[6] at $\beta \doteqdot \infty$.

5. RANDOM-SITE ISING MODEL

In the present section, we consider the random-site Ising model where the species μ_i of each site i is distributed by a probability distribution function $p(\mu_i)$, independently of other sites, and the exchange integral between a nearest-neighbor pair of sites of species μ and μ' is $J(\mu\mu')$. Now we have the average with respect to the probability density of the species $\{\mu_i\}$ on the right-hand side of (2.1). As a consequence, we have

$$\chi^{(4)} = -\tfrac{1}{2}\beta^3\sum_{\mu} p(\mu)g^{(\mu)}(4) \tag{5.1}$$

$$\begin{aligned} g^{(\mu)}(4) = 4\Big[& 1 + 4\sum_{\mu'} w(\mu\mu')p(\mu')f^{(\mu')}(1) + 6\sum_{\mu'} w_2(\mu\mu')p(\mu')f^{(\mu')}(2) \\ & + 4\sum_{\mu'} w_1(\mu\mu')p(\mu')f^{(\mu')}(3) \\ & + 3(B-1)\sum_{\mu_1}\sum_{\mu_2} w(\mu\mu_1)w(\mu\mu_2)p(\mu_1)p(\mu_2)f^{(\mu_1\mu_2)}(1,1) \\ & + 4(B-1)\sum_{\mu_1}\sum_{\mu_2} w(\mu\mu_1)w_2(\mu\mu_2)p(\mu_1)p(\mu_2)f^{(\mu_1\mu_2)}(2,1) \\ & + \cdots \Big] \end{aligned} \tag{5.2}$$

in place of (2.4) and (2.5), where

$$w(\mu\mu') = \tanh[\beta J(\mu, \mu')], \qquad w_\nu(\mu\mu') = w(\mu\mu')^\nu \tag{5.3}$$

(2.7) is replaced by

$$f^{(\mu)}(1) = 1 + \sum_{\mu'} w(\mu\mu')p(\mu')f^{(\mu')}(1) \tag{5.4a}$$

$$f^{(\mu_1\mu_2)}(1,1) = 2 + 2\sum_{\mu'} w(\mu_1\mu')p(\mu')f^{(\mu')}(1) + 2\sum_{\mu'} w(\mu_2\mu')p(\mu')f^{(\mu')}(1)$$
$$+ B\sum_{\mu_1'}\sum_{\mu_2'} w(\mu_1\mu_1')w(\mu_2\mu_2')p(\mu_1')p(\mu_2')f^{(\mu_1'\mu_2')}(1,1) \tag{5.4b}$$

$$f^{(\mu)}(2) = 1 + 2\sum_{\mu'} w(\mu\mu')p(\mu')f^{(\mu')}(1) + \tfrac{1}{2}(B-1)\sum_{\mu_1}\sum_{\mu_2} w(\mu\mu_1)w(\mu\mu_2)$$
$$\times p(\mu_1)p(\mu_2)f^{(\mu_1\mu_2)}(1,1) + \sum_{\mu'} w_2(\mu\mu')p(\mu')f^{(\mu')}(2) \tag{5.4c}$$

$$\cdots$$

The equations corresponding to (2.8) are obtained from these equations by using the solutions of the eigenvalue problems of the matrices $A^{(\nu)}$ of which $\mu\mu'$ elements are given by

$$A^{(\nu)}(\mu\mu') = p(\mu)^{1/2}w(\mu\mu')^{\nu}p(\mu')^{1/2} \tag{5.5}$$

The eigenvalues and eigenvectors of $A^{(\nu)}$ are denoted by $\kappa_\alpha^{(\nu)}(\beta)$ and $\phi_\alpha^{(\nu)}$:

$$A^{(\nu)}\phi_\alpha^{(\nu)} = \kappa_\alpha^{(\nu)}(\beta)\phi_\alpha^{(\nu)} \tag{5.6}$$

We write $\kappa_\alpha^{(1)}(\beta)$ and $\phi_\alpha^{(1)}$ simply as $\kappa_\alpha(\beta)$ and ϕ_α. We denote the elements of $\phi_\alpha^{(\nu)}$ as $\phi_\alpha^{(\nu)}(\mu)$, and normalize $\phi_\alpha^{(\nu)}$ so that $|\phi_\alpha^{(\nu)}|^2 = \sum_\mu \phi_\alpha^{(\nu)}(\mu)^2 = 1$, then we obtain

$$p(\mu)^{1/2}f^{(\mu)}(1) = \sum_\alpha \phi_\alpha(\mu)\frac{1}{1-\kappa_\alpha(\beta)}C_\alpha(1) \tag{5.7a}$$

$$C_\alpha(1) = (\phi_\alpha, p^{1/2}) \tag{5.8a}$$

$$p(\mu_1)^{1/2}p(\mu_2)^{1/2}f^{(\mu_1\mu_2)}(1,1)$$
$$= \sum_{\alpha_1}\sum_{\alpha_2}\phi_{\alpha_1}(\mu_1)\phi_{\alpha_2}(\mu_2)\frac{1}{1-B\kappa_{\alpha_1}(\beta)\kappa_{\alpha_2}(\beta)}C_{\alpha_1\alpha_2}(1,1) \tag{5.7b}$$

$$C_{\alpha_1\alpha_2}(1,1) = \sum_{\mu_1}\sum_{\mu_2}\phi_{\alpha_1}(\mu_1)\phi_{\alpha_2}(\mu_2)p(\mu_1)^{1/2}$$
$$\times\{\text{rhs of (5.4b), excluding the last term}\} \tag{5.8b}$$

$$p(\mu)^{1/2}f^{(\mu)}(2) = \sum_\alpha \phi_\alpha^{(2)}(\mu)\frac{1}{1-\kappa_\alpha^{(2)}(\beta)}C_\alpha(2) \tag{5.7c}$$

$$C_\alpha(2) = \sum_\mu \phi_\alpha^{(2)}(\mu)p(\mu)^{1/2}$$
$$\times\{\text{rhs of (5.4c), excluding the last term}\} \tag{5.8c}$$

$$\cdots$$

$(\phi_\alpha, p^{1/2})$ in (5.8a) denotes

$$(\phi_\alpha, p^{1/2}) \equiv \sum_\mu \phi_\alpha(\mu) p(\mu)^{1/2} \tag{5.9}$$

Solving these equations from the top equation and substituting the solutions into (5.1) and (5.2), we obtain an expression for $\chi^{(4)}$. In the expression, the factor

$$1/\left[1 - B^{k-1}\kappa_{\alpha_1}^{(\nu_1)}(\beta)\kappa_{\alpha_2}^{(\nu_2)}(\beta) \cdots \kappa_{\alpha_k}^{(\nu_k)}(\beta)\right] \tag{5.10}$$

appears in place of the factor $1/(1 - B^{k-1}w_{\nu_1}w_{\nu_2} \cdots w_{\nu_k})$. We can easily see that the factor which diverges at the highest temperature is either

$$1/\left[1 - B^3\kappa_1(\beta)^4\right] \quad \text{or} \quad 1/\left[1 - B\kappa_1^{(2)}(\beta)^2\right]$$

where $\kappa_1^{(\nu)}(\beta)$ is the eigenvalue of the largest absolute value among $\kappa_\alpha^{(\nu)}(\beta)$. In a preceding note,[9] we define $\beta_2(x)$ and $\beta_1(x)$ by denoting the positive solution β of $\kappa_1^{(2)}(\beta) = x$ by $\beta_2(x)$ and the smallest positive solution β of $|\kappa_1(\beta)| = x$ by $\beta_1(x)$. Then we obtain $T_c^{(4)}$ as (4.4)–(4.6).

For the susceptibility $\chi^{(2)}$, (3.2) is replaced by

$$\beta^{-1}\chi^{(2)} = \sum_\mu p(\mu) g^{(\mu)}(2) \tag{5.11}$$

$$g^{(\mu)}(2) = 1 + 2\sum_{\mu'} w(\mu\mu')p(\mu')f^{(\mu')}(1) + (B-1)\sum_{\mu_1}\sum_{\mu_2} w(\mu\mu_1)w(\mu\mu_2) \times p(\mu_1)p(\mu_2)f^{(\mu_1\mu_2)}(1,1) \tag{5.12}$$

By substituting (5.7a), (5.7b), and (5.8b), we obtain

$$\begin{aligned}\beta^{-1}\chi^{(2)} = {} & 1 + 2\sum_\alpha \frac{\kappa_\alpha(\beta)}{1-\kappa_\alpha(\beta)}(\phi_\alpha, p^{1/2})^2 + 2(B-1)\sum_{\alpha_1}\sum_{\alpha_2} \\ & \times\left[\sum_\mu p(\mu)^{1/2}\phi_{\alpha_1}(\mu)\phi_{\alpha_2}(\mu)\right]\frac{\kappa_{\alpha_1}(\beta)\kappa_{\alpha_2}(\beta)}{1-B\kappa_{\alpha_1}(\beta)\kappa_{\alpha_2}(\beta)} \\ & \times\left[(\phi_{\alpha_1}, p^{1/2})(\phi_{\alpha_2}, p^{1/2}) + (\phi_{\alpha_2}, p^{1/2})\frac{\kappa_{\alpha_1}(\beta)}{1-\kappa_{\alpha_1}(\beta)}\right. \\ & \left. + (\phi_{\alpha_1}, p^{1/2})\frac{\kappa_{\alpha_2}(\beta)}{1-\kappa_{\alpha_2}(\beta)}\right]\end{aligned} \tag{5.13}$$

The highest temperature at which this expression diverges is determined by $B\kappa_1(\beta)^2 = 1$, which gives $T_c^{(2)} = 1/k_B\beta_c^{(2)}$, where

$$\beta_c^{(2)} = \beta_1(B^{-1/2}) \tag{5.14}$$

The diluted-site Ising model is naturally treated by the method in this section. We then see that $\chi^{(4)}_{r\text{-}s}$ and $\chi^{(2)}_{r\text{-}s}$ for this model are equal to the products of the concentration of magnetic sites and $\chi^{(4)}_{r\text{-}b}$ and $\chi^{(2)}_{r\text{-}b}$, respectively, for a diluted-bond system, as stated at the end of the preceding section.

6. SUMMARY

An explicit expression for the fourth-order susceptibility $\chi^{(4)}$ is obtained for the Ising model on the Cayley tree. For the regular system it is given by (2.10) with (2.9) and $w = \tanh(J/k_B T)$. For the random-bond Ising system, $\chi^{(4)}$ is given by (2.4), (2.5), (2.8), (2.9), (4.1), and (4.2). The ordinary susceptibility $\chi^{(2)}$ for that system is given by (3.3) with (4.1). For the diluted-bond Ising model, we have (4.8) in place of (4.1) and (4.2), and $\chi^{(2)}$ is given by (4.12). The susceptibilities for the diluted-site Ising model are obtained by multiplying the concentration of magnetic sites to those for a corresponding diluted-bond Ising model. For the random-site Ising model, $\chi^{(2)}$ is given by (5.13). As for $\chi^{(4)}$ for this system, only an explanation is given of the way of obtaining the explicit expression. The temperatures at which $\chi^{(2n)}$ diverge are investigated for the random-bond and the random-site Ising model in separate notes.[8,9]

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APPENDIX: SOLUTION OF (2.8)

The solution of (2.8) obtained by taking account of the relations (2.6) is given by (2.9) and the following equations:

$$f(2) = \tfrac{1}{2} f(1,1)$$

$$f(2,1) = \tfrac{1}{2} f(1,1,1)$$

$$f(3) = \tfrac{1}{6}(1 + w + w^2) f(1,1,1)$$

$$f(2,1,1) = \tfrac{1}{2} f(1,1,1,1)$$

$$f(2,2) = \tfrac{1}{4} f(1,1,1,1)$$

$$f(3,1) = \frac{1}{6}\,\frac{1 - Bw^4}{1 - Bw^2} f(1,1,1,1) + \frac{2}{3}\,\frac{w(1 + w)}{1 - Bw^2} f(1,1,1)$$

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